10- Sinusoidal Steady-State Power Calculations

- All electrical energy is supplied in the form of sinusoidal voltages and currents.
- Our interest is the average power delivered to or supplied from a pair of terminals as a result of sinusoidal voltages and currents

**Instantaneous Power**

![Diagram](image)

Instantaneous Power is the power at any instant of time is

\[ p = vi \]

where

\[
\begin{align*}
    v &= V_m \cos(\omega t + \theta_v) \\
    i &= I_m \cos(\omega t + \theta_i)
\end{align*}
\]

Let's shift both voltage and current by \( \theta_i \)

\[
\begin{align*}
    v &= V_m \cos(\omega t + \theta_v - \theta_i) \\
    i &= I_m \cos(\omega t)
\end{align*}
\]

Therefore, instantaneous power becomes

\[
p = vi = V_m I_m \cos(\omega t + \theta_v - \theta_i) \cos \omega t \quad \text{(1.1)}
\]

To simplify this, we can use the following trigonometric identity

\[
\cos \alpha \cos \beta = \frac{1}{2} \cos(\alpha - \beta) + \frac{1}{2} \cos(\alpha + \beta)
\]

\[
p = \frac{V_m I_m}{2} \cos(\theta_v - \theta_i) + \frac{V_m I_m}{2} \cos(2\omega t + \theta_v - \theta_i) \quad \text{(1.2)}
\]

We know that

\[
\cos(\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta
\]

Therefore, instantaneous power

\[
p = \frac{V_m I_m}{2} \cos(\theta_v - \theta_i) + \frac{V_m I_m}{2} \cos(\theta_v - \theta_i) \cos 2\omega t - \frac{V_m I_m}{2} \sin(\theta_v - \theta_i) \sin 2\omega t \quad \text{(1.3)}
\]
In this equation,
- the first term is constant,
- the 2\textsuperscript{nd} and 3\textsuperscript{rd} show that instantaneous power has twice frequency of the voltage or current.
- It has negative for some portion of cycle. That means, energy stored in the inductors or capacitors is now being extracted.

**Average and Reactive Power.**

We can divide the instantaneous power (last equation) in three terms.

\[ p = P + P \cos 2\omega t - Q \sin 2\omega t \]  \hspace{1cm} (1.4)

where \( P \) is called the *average power or real power*

\[ P = \frac{V_m I_m}{2} \cos(\theta_v - \theta_i) \]  \hspace{1cm} [W]  \hspace{1cm} (1.5)

\( Q \) is called the *reactive power*

\[ Q = \frac{V_m I_m}{2} \sin(\theta_v - \theta_i) \]  \hspace{1cm} [VAR]  \hspace{1cm} (1.6)

The average power be represented in the following form too

\[ P = \frac{1}{T} \int_{t_s}^{t_s+T} p dt \]

\( T \) is period of the sinusoidal function.
Power for Purely Resistive Circuits

If the circuit between the terminal is purely resistive, there is no phase different between voltage and current, which means $\theta_v = \theta_i$, so that instantaneous power will be

$$p = \frac{V_m I_m}{2} (1 + \cos 2\omega t)$$

This is called as the instantaneous real power.
- It has some average value
- Frequency is $2\omega$
- Can never be negative, that means power can not be extracted from purely resistive circuit

Power for Purely Inductive Circuits

If the circuit between the terminal is purely inductive, there is phase different between voltage and current, The current lags the voltage by $90^\circ$ which means $\theta_i = \theta_v - 90^\circ$ ($\theta_v - \theta_i = +90^\circ$), so that instantaneous power will be

$$p = -\frac{V_m I_m}{2} \sin 2\omega t$$

This is called as the instantaneous reactive power.
- The average will be zero.
- Frequency is $2\omega$
- When $p$ is positive, energy is being stored in the magnetic fields associate with the inductive elements,
- When $p$ is negative, energy is being extracted from the magnetic fields.
Power for Purely Capacitive Circuit

If the circuit between the terminal is purely capacitive, there is phase different between voltage and current. The current leads the voltage by $90^\circ$ which means $\theta_i = \theta_v + 90^\circ$ ($\theta_v - \theta_i = -90^\circ$), so that instantaneous power will be

$$p = \frac{V_m I_m}{2} \sin 2\omega t$$

This is called as the instantaneous reactive power.

- The average will be zero.
- Frequency is $2\omega$.
- When $p$ is positive, energy is being stored in the capacitive elements.
- When $p$ is negative, energy is being extracted from the capacitive elements.
The Power factor

The angle $\theta_v - \theta_i$ is the power factor angle.

The power factor is

\[ pf = \cos(\theta_v - \theta_i) \]  \hspace{1cm} (1.7)

The reactive factor is

\[ rf = \sin(\theta_v - \theta_i) \]

Depends on the phase, the current lags the voltage by 90°\(^\circ\),

we call lagging power factor \( pf = \cos(\theta_v - \theta_i) \)

and the current leads the voltage by 90°\(^\circ\),

we call leading power factor \( pf = \cos(\theta_i - \theta_v) \)
**Example:**

\[ v = 100 \cos(\omega t + 15^\circ) \text{ V.} \]
\[ i = 4 \sin(\omega t - 15^\circ) \text{ A} \]

Let's convert current \( i(t) \) in cos function

\[ i = 4 \cos(\omega t - 15^\circ - 90^\circ) = 4 \cos(\omega t - 105^\circ) \]

the average power or real power will be

\[ P = \frac{V_m I_m}{2} \cos(\theta_v - \theta_i) \]
\[ P = \frac{(100)(4)}{2} \cos(15 - (-105)) = -100 \text{ W} \]

(Network inside the box is delivering average power to the terminal)

The reactive power

\[ Q = \frac{V_m I_m}{2} \sin(\theta_v - \theta_i) \]
\[ Q = \frac{(100)(4)}{2} \sin(15 - (-105)) = 173.21 \text{ VAR} \]

(Inside the box is absorbing magnetizing vars at its terminal.)

The instantaneous power will be

\[ p = -100 - 100 \cos 2\omega t - 173.21 \sin 2\omega t \]
The RMS Value and Power Calculation

The RMS value of a periodic sinusoidal function is defined as a square root of the mean value of the square sinusoidal function. Also, the RMS value can be called as effective value of the

Ex: The RMS value of

\[ v(t) = V_m \cos(\omega t + \theta) \]

is

\[
(1.8) V_{rms} = \sqrt{\frac{1}{T} \int_{0}^{T} V_m^2 \cos^2(\omega t + \theta) dt} = \frac{V_m}{\sqrt{2}} 
\]

Let's look at the average power delivered to the resistor

\[ P = \frac{V_{rms}^2}{R} = \frac{V_m^2}{2R} \]

If we know the current on the resistor, we can find the average power delivered to the resistor as

\[ P = I_{rms}^2 R = \frac{I_m^2}{2} R \]

The previous pages we give the average power as

\[ P = \frac{V_m I_m}{2} \cos(\theta_v - \theta_i) \quad (1.10) \]

This can be written in terms of RMS value

\[ P = \frac{V_m I_m}{\sqrt{2}} \cos(\theta_v - \theta_i) \]

\[ P = V_{rms} I_{rms} \cos(\theta_v - \theta_i) \quad (1.11) \]
The reactive power

\[ Q = \frac{V_m I_m \sin(\theta_v - \theta_i)}{2} \]
\[ Q = V_{rms} I_{rms} \sin(\theta_v - \theta_i) \]

**Example:**

\[ V_{rms} = 110 \, V, \quad R = 50 \, \Omega, \]

Find \( V_m \) and the Average power delivered to the resistor

\[ V_{rms} = \frac{V_m}{\sqrt{2}} \]
\[ \Rightarrow V_m = V_{rms} \sqrt{2} = 110 \sqrt{2} = 155.56 \, V \]
\[ P = \frac{V_{rms}^2}{R} = \frac{110^2}{50} = 242 \, W \]